1.1.1

We have

\[ \nabla f(x, y) = \begin{pmatrix} 2x + \beta y + 1 \\ 2y + \beta x + 2 \end{pmatrix} \]

Setting \( \nabla f(x, y) = 0 \), we obtain the system of equations

\[ \begin{pmatrix} 2 & \beta \\ \beta & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} 1 \\ 2 \end{pmatrix}. \]

This system has a unique solution (a unique stationary point) except when \( \beta^2 = 4 \).

If \( \beta^2 = 4 \), it can be verified that there is no solution to the above system (no stationary point).

Assuming \( \beta^2 \neq 4 \), for the stationary point to be a local minimum, the Hessian matrix of \( f \), which is

\[ Q = \begin{pmatrix} 2 & \beta \\ \beta & 2 \end{pmatrix}, \]

must be positive semidefinite. But if this is so, \( f(x, y) \) will be a convex quadratic function and each local minimum will be global.

The Hessian \( Q \) will be positive definite if and only if \( \beta^2 < 4 \) and positive semidefinite if \( \beta^2 = 4 \), in which case there is no stationary point by the preceding discussion.

Thus, if \( \beta^2 < 4 \), there is a unique stationary point which is a global minimum. If \( \beta^2 = 4 \), there is no stationary point. If \( \beta^2 > 4 \), there is a unique stationary point which, however, is not a local minimum.

1.1.2 (b)

We have

\[ \nabla f(x, y) = \begin{pmatrix} x + \cos y \\ -x \sin y \end{pmatrix}, \quad \nabla^2 f(x, y) = \begin{pmatrix} 1 & -\sin y \\ -\sin y & -x \cos y \end{pmatrix} \]

Thus the stationary points of \( f \) are:

\[ \{((-1)^{k+1}, k\pi) \mid k = \text{integer}\}, \quad \{(0, k\pi + \pi/2) \mid k = \text{integer}\}. \]
Of these, the local minima are
\[ \{((-1)^{k+1}, k\pi) \mid k = \text{integer}\}. \]

1.1.3
(a) Since the function \( f(x^* + \alpha d) \) is minimized at \( \alpha = 0 \) for all \( d \in \mathbb{R}^n \), we have for all \( \alpha \) and \( i \)

\[ f(x^* + \alpha e_i) \geq f(x^*), \]

which implies that
\[
\lim_{\alpha \to 0^+} \frac{f(x^* + \alpha e_i) - f(x^*)}{\alpha} \geq 0, \quad \lim_{\alpha \to 0^-} \frac{f(x^* + \alpha e_i) - f(x^*)}{\alpha} \leq 0,
\]
or
\[
\left( \frac{\partial f(x^*)}{\partial x_i} \right) = 0, \quad \forall \ i.
\]

(b) Consider the function \( f(y, z) = (z - py^2)(z - qy^2) \), where \( 0 < p < q \) and let \( x^* = (0, 0) \).

We first show that \( g(\alpha) \) is minimized at \( \alpha = 0 \) for all \( d \in \mathbb{R}^2 \). We have

\[ g(\alpha) = f(x^* + \alpha d) = f(\alpha d) = (\alpha d_2 - p\alpha^2 d_1^2)(\alpha d_2 - q\alpha^2 d_1^2) = \alpha^2(d_2 - p\alpha d_1^2)(d_2 - q\alpha d_1^2). \]

Also,
\[
g'(\alpha) = 2\alpha(d_2 - p\alpha d_1^2)(d_2 - q\alpha d_1^2) + \alpha^2(-pd_1^2)(d_2 - q\alpha d_1^2) + \alpha^2(d_2 - p\alpha d_1^2)(-qd_1^2).
\]

Thus \( g'(0) = 0 \). Furthermore,
\[
g''(\alpha) = 2(d_2 - p\alpha d_1^2)(d_2 - q\alpha d_1^2) + 2\alpha(-pd_1^2)(d_2 - q\alpha d_1^2) + 2\alpha(d_2 - p\alpha d_1^2)(-qd_1^2)
+ 2\alpha(d_2 - p\alpha d_1^2)(-qd_1^2) + \alpha^2(-pd_1^2)(-qd_1^2)
+ 2\alpha(d_2 - p\alpha d_1^2)(-qd_1^2) + \alpha^2(-pd_1^2)(-qd_1^2).
\]

Thus \( g''(0) = 2d_2^2 \), which is greater than 0 if \( d_2 \neq 0 \). If \( d_2 = 0 \), \( g(\alpha) = pq\alpha^4 d_1^4 \), which is clearly minimized at \( \alpha = 0 \).

Therefore, \((0, 0)\) is a local minimum of \( f \) along every line that passes through \((0, 0)\).

Let’s now show that if \( p < m < q \), \( f(y, my^2) < 0 \) if \( y \neq 0 \) and that \( f(y, my^2) \geq 0 \) otherwise. Consider a point of the form \((y, my^2)\). We have \( f(y, my^2) = y^4(m - p)(m - q) \).

Clearly, \( f(y, my^2) < 0 \) if and only if \( p < m < q \) and \( y \neq 0 \). In any \( \epsilon \)-neighborhood of \((0, 0)\), there exists a \( y \neq 0 \) such that for some \( m \in (p, q) \), \((y, my^2)\) also belongs to the neighborhood. Since \( f(0, 0) = 0 \), we see that \((0, 0)\) is not a local minimum.

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