Recitation 2 Solutions

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1.3.1

The Hessian matrix of the function \( f \) is

\[
Q = \begin{pmatrix}
2 & 1.999 \\
1.999 & 2
\end{pmatrix}
\]

which has largest and smallest eigenvalues \( M = 3.999 \) and \( m = 0.001 \) respectively. Hence

\[
\frac{f(x_{k+1})}{f(x_k)} \leq \left( \frac{3.998}{4} \right)^2 \approx 0.999
\]

Let \( v_m \) and \( v_M \) be the normalized eigenvectors of \( Q \) (see Prop. A.17, Appendix A) corresponding to \( m \) and \( M \), respectively, and let

\[
x^0 = \frac{s}{m} v_m \pm \frac{s}{M} v_M, \quad s \in \mathbb{R},
\]

(cf. Fig. 1.3.2 in Section 1.3). We have

\[
x^1 = x^0 - \alpha^0 Q x^0 = \left( \frac{1}{m} - \alpha^0 \right) s v_m \mp \left( \frac{1}{M} - \alpha^0 \right) s v_M
\]

and

\[
f(x^1) = s^2 \left[ m \left( \frac{1}{m} - \alpha^0 \right)^2 + M \left( \frac{1}{M} - \alpha^0 \right)^2 \right]
\]

Using the line minimization stepsize rule, i.e., a stepsize

\[
\alpha^* = \arg \min_{\alpha} \left\{ s^2 \left[ m \left( \frac{1}{m} - \alpha^0 \right)^2 + M \left( \frac{1}{M} - \alpha^0 \right)^2 \right] \right\} = \frac{2}{m + M}
\]

, we get the first iteration,

\[
x_1 = \left( \frac{M - m}{M + m} \right) \left( \frac{s}{m} v_m \mp \frac{s}{M} v_M \right) ,
\]

which has the same form as \( x_0 \) except for the factor \( \frac{M - m}{M + m} \). Hence starting the iterations with \( x_0 \) we have for all \( k \)

\[
\frac{f(x_{k+1})}{f(x_k)} = \left( \frac{M - m}{M + m} \right)^2 .
\]
1.5.2

We have
\[ x^{k+1} = y^k - \alpha(y^k - z_2) \]
\[ = x^k - \alpha(x^k - z_1) - \alpha(x^k - \alpha(x^k - z_1) - z_2) \]
\[ = x^k(1 - 2\alpha + \alpha^2) + \alpha[(1 - \alpha)z_1 + z_2] = ax^k + b, \]
where \( a = 1 - 2\alpha + \alpha^2 \) and \( b = \alpha[(1 - \alpha)z_1 + z_2] \). Note that since \( 0 < \alpha < 1, 0 < a < 1 \).

Further expanding \( x^{k+1} \) yields
\[ x^{k+1} = a(ax^{k-1} + b) + b = a^2x^{k-1} + b(1 + a) \]
\[ = a^2(ax^{k-2} + b) + b(1 + a) = a^3x^{k-2} + b(1 + a + a^2) \]
\[ = \ldots + a^{k+1}x^0 + b(1 + a + \ldots + a^k) \]
So
\[ \lim_{k \to \infty} x^k = 0 + b \frac{1}{1 - a} = \frac{(1 - \alpha)z_1 + z_2}{2 - \alpha} = x(\alpha). \]

Similarly for \( y \), we have
\[ y^{k+1} = x^{k+1} - \alpha(x^{k+1} - z_1) = y^k - \alpha(y^k - z_2) - \alpha(y^k - \alpha(y^k - z_2) - z_1), \]
and so \( y_{k+1} \) is related analogously to \( y_k \) as \( x_{k+1} \) is to \( x_k \). Therefore we have
\[ \lim_{k \to \infty} y^k = \frac{(1 - \alpha)z_2 + z_1}{2 - \alpha} = y(\alpha). \]

From these expressions, it is clear that unless \( z_1 = z_2 \) or \( \alpha = 0 \), \( x(\alpha) \neq y(\alpha) \), and neither is equal to the optimal least squares solution \( x^* = (z_1 + z_2)/2 \). However, we do have \( x(\alpha) \to x^* \) and \( y(\alpha) \to x^* \) as \( \alpha \to 0 \).

2.1.6

The problem is equivalent to
\[
\min_{-a_1 \leq x_i \leq a_n} \sum_{i=1}^n x_i = 1, \quad x_i \geq 0, \quad \forall i.
\]
From the discussion in Example 2.1.2, the necessary optimality conditions are
\[ x^*_i > 0 \quad \Rightarrow \quad \frac{\partial f(x^*)}{\partial x_i} \leq \frac{\partial f(x^*)}{\partial x_j}, \quad \forall j \]
or
\[ -a_i(x^*_i)^{a_i-1} \prod_{k \neq i}(x^*_k)^{a_k} \leq -a_j(x^*_j)^{a_j-1} \prod_{k \neq j}(x^*_k)^{a_k}, \quad \forall j \]
or
\[ a_i x^*_i \geq a_j x^*_j, \quad \forall j. \]
It is clear that if $x^*$ is a global minimum, we must have $x^*_i > 0$ for all $i$. Therefore, the above relation is equivalent to
\[ a_i x^*_j = a_j x^*_i, \quad \forall \ i, j. \]
Summing over all $j$ and using the constraint $\sum_j x^*_j = 1$, we have
\[ \sum_j a_i x^*_j = \sum_j a_j x^*_i \]
or
\[ a_i \sum_j x^*_j = x^*_i \sum_j a_j \]
or
\[ x^*_i = \frac{a_i}{\sum_j a_j}, \quad \forall \ i. \]
In fact, this is the only point satisfying the necessary conditions. Since the constraint region is compact and the cost function is continuous, a global maximum exists by Weierstrass’ theorem, and thus this point is the unique global maximum.