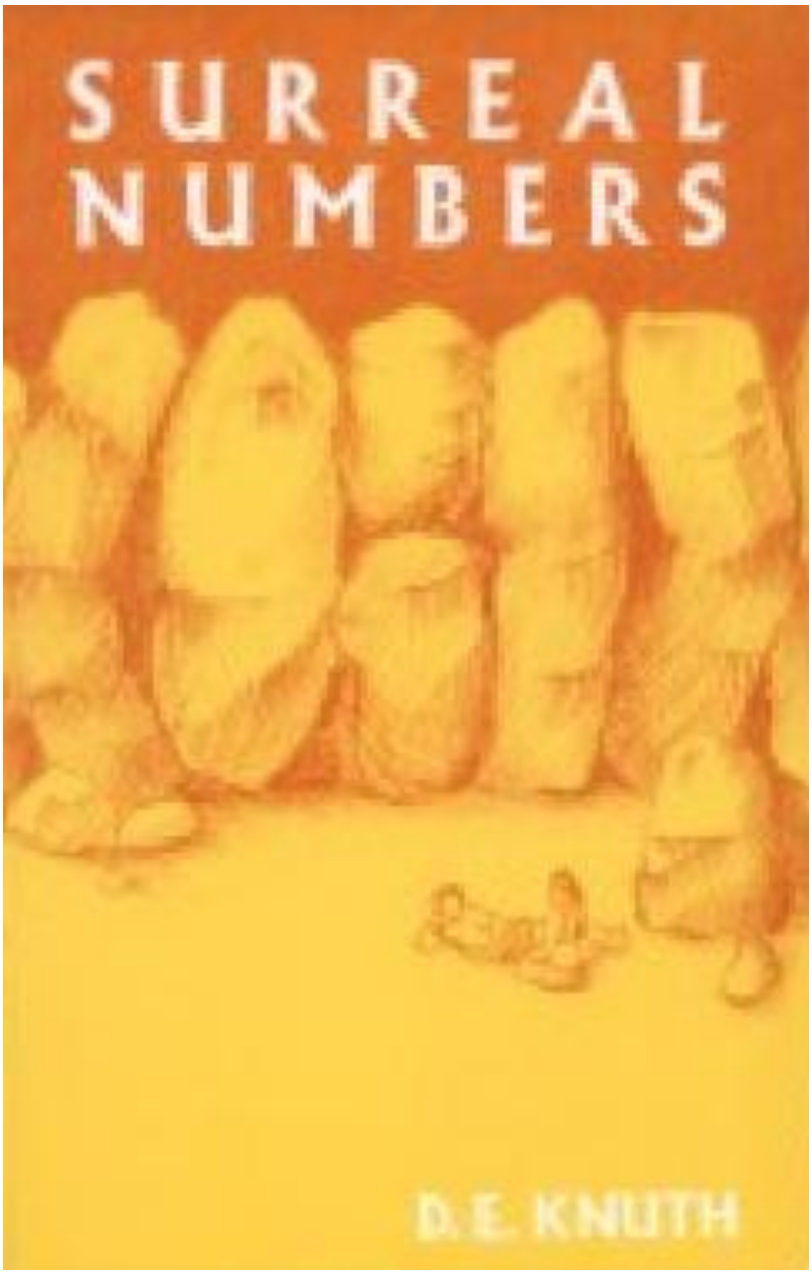
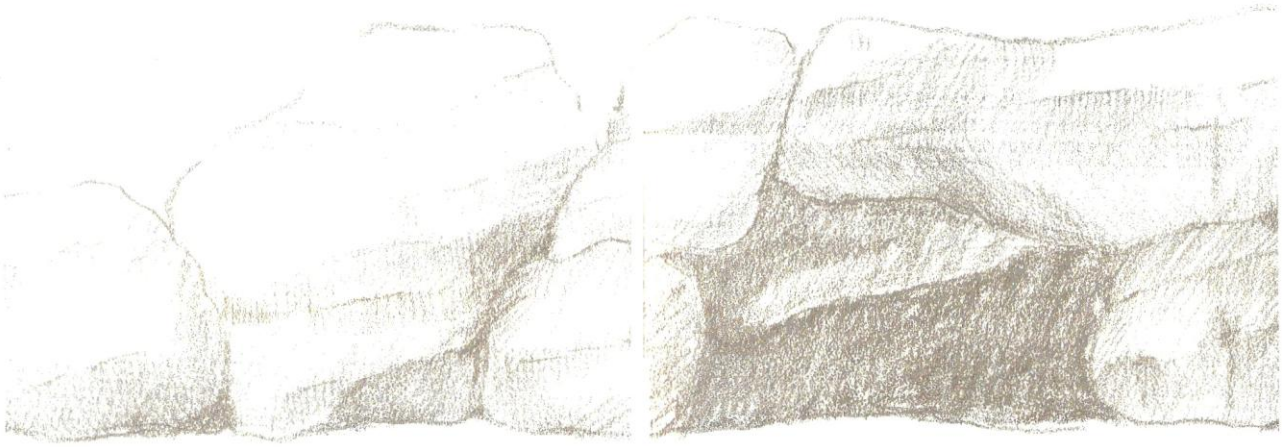

SURREAL
NUMBERS
D. E. KNUTH

1 THE ROCK



A. Bill, do you think you've found yourself?

B. What?

A. I mean—here we are on the edge of the Indian Ocean, miles away from civilization. It's been months since we ran off to avoid getting swept up in the system, and "to find ourselves." I'm just wondering if you think we've done it.

B. Actually, Alice, I've been thinking about the same thing. These past months together have been really great—we're completely free, we know each other, and we feel like real people again instead of like machines. But lately I'm afraid I've been missing some of the things we've "escaped" from. You know, I've got this fantastic craving for a book to read—*any* book, even a textbook, even a math textbook. It sounds crazy, but I've been lying here wishing I had a crossword puzzle to work on.

A. Oh, c'mon, not a crossword puzzle; that's what your *parents* like to do. But I know what you mean, we need some mental stimulation. It's kinda like the end of summer vacations when we were kids. In May every year we couldn't wait to get out of school, and the days simply dragged on until vacation started, but by September we were real glad to be back in the classroom.

B. Of course, with a loaf of bread, a jug of wine, and thou beside me, these days aren't exactly "dragging on." But I think maybe the most important thing I've learned on this trip is that the simple, romantic life isn't enough for me. I need something complicated to think about.

A. Well, I'm sorry I'm not complicated enough for you. Why don't we get up and explore some more of the beach? Maybe we'll find some pebbles or something that we can use to make up some kind of a game.

B. (sitting up) Yeah, that's a good idea. But first I think I'll take a little swim.

A. (running toward the water) Me, too—bet you can't catch me!

. . . . . .

B. Hey, what's that big black rock half-buried in the sand over there?

A. Search me, I've never seen anything like it before. Look, it's got some kind of graffiti on the back.

B. Let's see. Can you help me dig it out? It looks like a museum piece. Unnh! Heavy, too. The carving might be some old Arabian script...no, wait, I think it's maybe Hebrew; let's turn it around this way.

A. Hebrew! Are you sure?

B. Well, I learned a lot of Hebrew when I was younger, and I can almost read this....

A. I heard there hasn't been much archaeological digging around these parts. Maybe we've found another Rosetta Stone or something. What does it say, can you make anything out?

B. Wait a minute, gimme a chance.... Up here at the top right is where it starts, something like "In the beginning everything was void, and...."

A. Far out! That sounds like the first book of Moses, in the Bible. Wasn't he wandering around Arabia for forty years with his followers before going up to Israel? You don't suppose....

B. No, no, it goes on much different from the traditional account. Let's lug this thing back to our camp, I think I can work out a translation.

A. Bill, this is wild, just what you needed!

B. Yeah, I did say I was dying for something to read, didn't I. Although this wasn't exactly what I had in mind! I can hardly wait to get a good look at it—some of the things are kinda strange, and I can't figure out whether it's a story or what. There's something about numbers, and....

A. It seems to be broken off at the bottom; the stone was originally longer.

B. A good thing, or we'd never be able to carry it. Of course it'll be just our luck to find out that the message is getting interesting, right when we come to the broken place.

A. Here we are. I'll go pick some dates and fruit for supper while you work out the translation. Too bad languages aren't my thing, or I'd try to help you.

. . . . . .

B. Okay, Alice, I've *got* it. There are a few doubtful places, a couple signs I don't recognize; you know, maybe some obsolete word forms. Overall I think I know what it says, though I don't know what it means. Here's a fairly literal translation:

> **In the beginning, everything was void, and J. H. W. H. Conway began to create numbers. Conway said, "Let there be two rules which bring forth all numbers large and small. This shall be the first rule: Every number corresponds to two sets of previously created numbers, such that no member of the left set is greater than or equal to any member of the right set. And the second rule shall be this: One number is less than or equal to another number if and only if no member of the first number's left set is greater than or equal to the second number, and no member of the second number's right set is less than or equal to the first number." And Conway examined these two rules he had made, and behold! they were very good.**
>
> **And the first number was created from the void left set and the void right set. Conway called this number "zero,"**

> and said that it shall be a sign to separate positive numbers from negative numbers. Conway proved that zero was less than or equal to zero, and he saw that it was good. And the evening and the morning were the day of zero. On the next day, two more numbers were created, one with zero as its left set and one with zero as its right set. And Conway called the former number "one," and the latter he called "minus one." And he proved that minus one is less than but not equal to zero and zero is less than but not equal to one. And the evening. . .

That's where it breaks off.

A. Are you *sure* it reads like that?

B. More or less. I dressed it up a bit.

A. But "Conway". . .that's not a Hebrew name. You've got to be kidding.

B. No, honest. Of course the old Hebrew writing doesn't show any vowels, so the real name might be Keenawu or something; maybe related to the Khans? I guess not. Since I'm translating into English, I just used an English name. Look, here are the places where it shows up on the stone. The J. H. W. H. might also stand for "Jehovah."

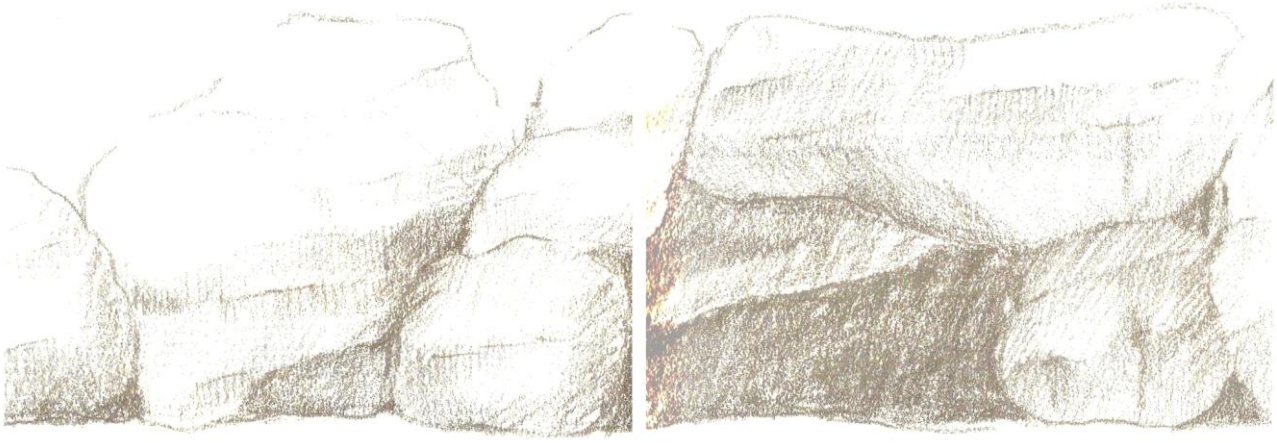
A. No vowels, eh? So it's real. . . But what do you think it means?

B. Your guess is as good as mine. These two crazy rules for numbers. Maybe it's some ancient method of arithmetic that's been obsolete since the wheel was invented. It might be fun to figure them out, tomorrow; but the sun's going down pretty soon so we'd better eat and turn in.

A. Okay, but read it to me once more. I want to think it over, and the first time I didn't believe you were serious.

B. (pointing) "In the beginning,. . ."

2 SYMBOLS



A. I think your Conway Stone makes sense after all, Bill. I was thinking about it during the night.

B. So was I, but I dozed off before getting anywhere. What's the secret?

A. It's not so hard, really; the trouble is that it's all expressed in words. The same thing can be expressed in symbols and then you can see what's happening.

B. You mean we're actually going to use the New Math to decipher this old stone tablet.

A. I hate to admit it, but that's what it looks like. Here, the first rule says that every number x is really a pair of sets, called the left set x_L and the right set x_R:

$$x = (x_L, x_R).$$

B. Wait a sec, you don't have to draw in the sand, I think we still have a pencil and some paper in my backpack. Just a minute... Here, use this.

A. $$x = (x_L, x_R).$$

These x_L and x_R are not just numbers, they're *sets* of numbers; and each number in the set is itself a pair of sets, and so on.

B. Hold it, your notation mixes me up. I don't know what's a set and what's a number.

A. Okay, I'll use capital letters for sets of numbers and small letters for numbers. Conway's first rule is that

$$x = (X_L, X_R), \qquad \text{where} \qquad X_L \ngeq X_R. \tag{1}$$

This means if x_L is any number in X_L and if x_R is any number in X_R, they must satisfy $x_L \ngeq x_R$. And that means x_L is not greater than or equal to x_R.

B. (scratching his head) I'm afraid you're still going too fast for me. Remember, you've already got this thing psyched out, but I'm still at the beginning. If a number is a pair of sets of numbers, each of which is a pair of sets of numbers, and so on and so on, how does the whole thing get started in the first place?

A. Good point, but that's the whole beauty of Conway's scheme. Each element of X_L and X_R must have been created previously, but on the first day of creation there weren't any previous numbers to work with; so both X_L and X_R are taken as the empty set!

B. I never thought I'd live to see the day when the empty set was meaningful. That's really creating something out of nothing, eh? But is $X_L \ngeq X_R$ when X_L and X_R are both equal to the empty set? How can you have something unequal itself?

Oh yeah, yeah, that's okay since it means no *element* of the empty set is greater than or equal to any element of the empty set—it's a true statement because there *aren't* any elements in the empty set.

A. So everything gets started all right, and that's the number called zero. Using the symbol $\emptyset$ to stand for the empty set, we can write

$$0 = (\emptyset, \emptyset).$$

B. Incredible.

A. Now on the second day, it's possible to use 0 in the left or right sets, so Conway gets two more numbers

$$-1 = (\emptyset, \{0\}) \qquad \text{and} \qquad 1 = (\{0\}, \emptyset).$$

B. Let me see, does this check out? For -1 to be a number, it has to be true that no element of the empty set is greater than or equal to 0. And for 1, it must be that 0 is not greater than any element of the empty set. Man, that empty set sure gets around! Someday I think I'll write a book called *Properties of the Empty Set*.

A. You'd never finish.

If X_L or X_R is empty, the condition $X_L \ngeq X_R$ is true no matter *what* is in the other set. This means that infinitely many numbers are going to be created.

B. Okay, but what about Conway's second rule?

A. That's what you use to tell whether $X_L \ngeq X_R$, when both sets are nonempty; it's the rule defining less-than-or-equal. Symbolically,

$$x \leq y \qquad \text{means} \qquad X_L \ngeq y \qquad \text{and} \qquad x \ngeq Y_R. \tag{2}$$

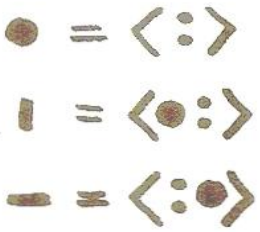
B. Wait a minute, you're way ahead of me again. Look, X_L is a set of numbers, and y is a number, which means a pair of sets of numbers. What do you mean when you write $X_L \ngeq y$?

A. I mean that every element of X_L satisfies $x_L \ngeq y$. In other words, no element of X_L is greater than or equal to y.

B. Oh, I see, and your rule (2) says also that x is not greater than or equal to any element of Y_R. Let me check that with the text...

A. The Stone's version is a little different, but $x \leq y$ must mean the same thing as $y \geq x$.

B. Yeah, you're right. Hey, wait a sec, look here at these carvings off to the side:



These are the symbols I couldn't decipher yesterday, and your notation makes it all crystal clear! Those double dots separate the left set from the right set. You must be on the right track.

A. Wow, equal signs and everything! That stone-age carver must have used ▬ to stand for -1; I almost like his notation better than mine.

B. I bet we've underestimated primitive people. They must have had complex lives and a need for mental gymnastics, just like us—at least when they didn't have to fight for food and shelter. We always oversimplify history when we look back.

A. Yes, but otherwise how could we look back?

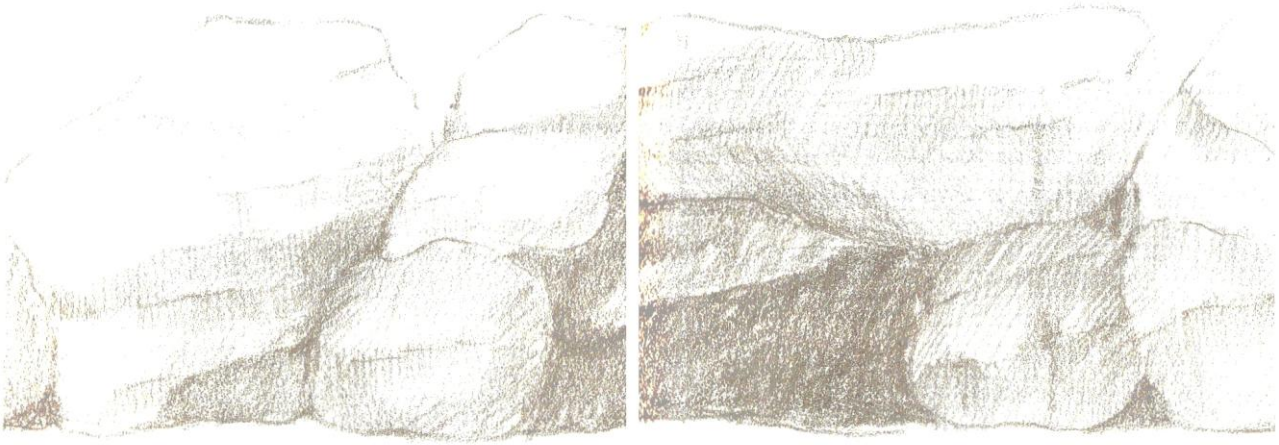
B. I see your point.

A. Now comes the part of the text I don't understand. On the first day of creation, Conway "proves" that $0 \leq 0$. Why should he bother to prove that something is less than or equal to itself, since it's obviously equal to itself. And then on the second day he "proves" that -1 is not equal to 0; isn't that obvious without proof, since -1 is a different number?

B. Hmm. I don't know about you, but I'm ready for another swim.

A. Good idea. That surf looks good, and I'm not used to so much concentration. Let's go!

3 PROOFS



B. An idea hit me while we were paddling around out there. Maybe my translation *isn't* correct.

A. What? It *must* be okay, we've already checked so much of it out.

B. I know; but now that I think of it, I wasn't quite sure of that word I translated "equal to." Maybe it has a weaker meaning,

"similar to" or "like." Then Conway's second rule becomes "One number is less than or *like* another number if and only if..." And later on, he proves that zero is less than or *like* zero, minus one is less than but not like zero, and so forth.

A. Oh, right, that must be it, he's using the word in an abstract technical sense that must be defined by the rules. So of *course* he wants to prove that 0 is less than or like 0, in order to see that his definition makes a number "like" itself.

B. So does his proof go through? By rule (2), he must show that no element of the empty set is greater than or like 0, and that 0 is not greater than or like any element of the empty set... Okay, it works, the empty set triumphs again.

A. More interesting is how he could prove that -1 is *not* like 0. The only way I can think of is that he proved that 0 is not less-than-or-like -1. I mean, we have rule (2) to tell whether one number is less than or like another; and if x is not less-than-or-like y, it isn't less than y and it isn't like y.

B. I see, we want to show that $0 \leq -1$ is false. This is rule (2) with $x = 0$ and $Y_R = \{0\}$, so $0 \leq -1$ if and only if $0 \ngeq 0$. But 0 *is* ≥ 0, we know that, so $0 \nleq -1$. He was right.

A. I wonder if Conway also tested -1 against 1; I suppose he did, although the rock doesn't say anything about it. If the rules are any good, it should be possible to prove that -1 is less than 1.

B. Well, let's see: -1 is $(\emptyset, \{0\})$ and 1 is $(\{0\}, \emptyset)$, so once again the empty set makes $-1 \leq 1$ by rule (2). On the other hand, $1 \leq -1$ is the same as saying that $0 \ngeq -1$ and $1 \ngeq 0$, according to rule (2), but we know that both of these are false. Therefore $1 \nleq -1$, and it must be that $-1 < 1$. Conway's rules seem to be working.

A. Yes, but so far we've been using the empty set in almost every argument, so the full implications of the rules aren't clear yet. Have you noticed that almost everything we've proved so far can be put into a framework like this: "If X and Y are any sets of numbers, then $x = (\emptyset, X)$ and $y = (Y, \emptyset)$ are numbers, and $x \leq y$."

B. It's neat the way you've just proved infinitely many things, by looking at the pattern I used in only a couple of cases. I guess that's what they call abstraction, or generalization, or something. But can you also prove that your x is strictly *less* than y? This was true in all the simple cases and I bet it's true in general.

A. Uh huh... Well no, not when X and Y are both empty, since that would mean $0 \ngeq 0$. But otherwise it looks very interesting. Let's look at the case when X is the empty set, and Y is not empty; is it true that 0 is less than $(Y, \emptyset)$?

B. If so, then I'd call $(Y, \emptyset)$ a "positive" number. That must be what Conway meant by zero separating the positive and negative numbers.

A. Yes, but look. According to rule (2), we will have $(Y, \emptyset) \leq 0$ if and only if no member of Y is greater than or like 0. So if, for example, Y is the set $\{-1\}$, then $(Y, \emptyset) \leq 0$. Do you want positive numbers to be ≤ 0?

Too bad I didn't take you up on that bet.

B. Hmm. You mean $(Y, \emptyset)$ is going to be positive only when Y contains some number that is zero or more. I suppose you're right. But at least we now understand everything that's on the stone.

A. Everything up to where it's broken off.

B. You mean...?

A. I wonder what happened on the *third* day.

B. Yes, we should be able to figure that out, now that we know the rules. It might be fun to work out the third day, after lunch.

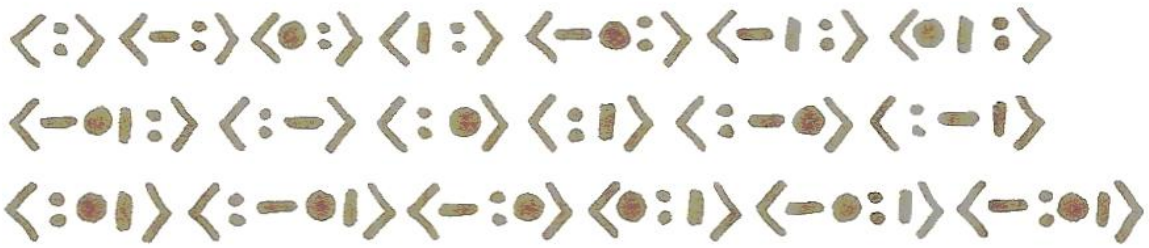
A. You'd better go catch some fish; our supply of dried meat is getting kinda low. I'll go try and find some coconuts.

4 BAD NUMBERS

B. I've been working on that Third Day problem, and I'm afraid it's going to be pretty hard. When more and more numbers have been created, the number of possible sets goes up fast. I bet that by the seventh day, Conway was ready for a rest.

A. Right. I've been working on it too and I get seventeen numbers on the third day.

B. Really? I found nineteen; you must have missed two. Here's my list:



A. I see you're using the Stone's notation. But why did you include  ? That was created already on the first day.

B. Well, we have to test the new numbers against the old, in order to see how they fit in.

A. But I only considered *new* numbers in my list of seventeen, so there must actually be *twenty* different at the end of the third day. Look, you forgot to include

in your list.

B. (blinking) So I did. Hmm . . . 20 by 20, that's 400 different cases we'll have to consider in rule (2). A lot of work, and not much fun either. But there's nothing else to do, and I know it'll bug me until I know the answer.

A. Maybe we'll think of some way to simplify the job once we get started.

B. Yeah, that would be nice . . .

Well, I've got one result, 1 is less than $(\{1\}, \emptyset)$. First I had to prove that $0 \ngeq (\{1\}, \emptyset)$.

A. I've been trying out a different approach. Rule (2) says we have to test every element of X_L to see that it isn't greater

than or like y, but it shouldn't be necessary to make so many tests. If any element of X_L is $\geq y$, then the *largest* element of X_L ought to be $\geq y$. Similarly, we need only test x against the *smallest* element of Y_R.

B. Yeah, that oughta be right . . . I can prove that 1 is less than $(\{0, 1\}, \emptyset)$ just like I proved it was less than $(\{1\}, \emptyset)$; the extra "0" in X_L didn't seem to make any difference.

A. If what I said is true, it will save us a lot of work, because each number (X_L, X_R) will behave in all $\leq$ relations exactly as if X_L were replaced by its largest element and X_R by its smallest. We won't have to consider any numbers in which X_L or X_R have two or more elements; ten of those twenty numbers in the list will be eliminated!

B. I'm not sure I follow you, but how on earth can we prove such a thing?

A. What we seem to need is something like this:

$$\text{if} \quad x \leq y \quad \text{and} \quad y \leq z, \qquad \text{then} \qquad x \leq z. \qquad \text{(T1)}$$

I don't see that this follows immediately, although it is consistent with everything we know.

B. At any rate, it ought to be true, if Conway's numbers are to be at all decent. We could go ahead and assume it, but it would be neat to show once and for all that it was true, just by using Conway's rules.

A. Yes, and we'd be able to solve the Third Day puzzle without much more work. Let's see, how can it be proved . . . .

B. Blast these flies! Just when I'm trying to concentrate. Alice, can you—no, I guess I'll go for a little walk.

. . . . . .

Any progress?

A. No, I seem to be going in circles, and the $\ngeq$ versus $\leq$ is confusing. Everything is stated negatively and things get incredibly tangled up.

B. Maybe (T1) isn't true.

A. But it *has* to be true. Wait, that's it! We'll try to *disprove* it. And when we fail, the cause of our failure will be a proof!

B. Sounds good—it's always easier to prove something wrong than to prove it right.

A. Suppose we've got three numbers x, y, and z for which

$$x \leq y, \qquad \text{and} \qquad y \leq z, \qquad \text{and} \qquad x \nleq z.$$

What does rule (2) tell us about "bad numbers" like this?

B. It says that

$$\begin{aligned} & X_L \ngeq y, \\ \text{and} \quad & x \ngeq Y_R, \\ \text{and} \quad & Y_L \ngeq z, \\ \text{and} \quad & y \ngeq Z_R, \end{aligned}$$

and then also $x \nleq z$, which means what?

A. It means one of the two conditions fails. Either there is a number x_L in X_L for which $x_L \geq z$, or there is a number z_R in Z_R for which $x \geq z_R$. With all these facts about x, y, and z, we ought to be able to prove *something*.

B. Well, since x_L is in X_L, it can't be greater than or equal to y. Say it's less than y. But $y \leq z$, so x_L must be . . . no, sorry, I can't use facts about numbers we haven't proved.

Going the other way, we know that $y \leq z$ and $z \leq x_L$ and $y \nleq x_L$; so this gives us three more bad numbers, and we can get more facts again. But that looks hopelessly complicated.

A. Bill! You've got it.

B. Have I?

A. If (x, y, z) are three bad numbers, there are two possible cases.

Case 1, some $x_L \geq z$: Then (y, z, x_L) are three more bad numbers.

Case 2, some $z_R \leq x$: Then (z_R, x, y) are three more bad numbers.

B. But aren't you still going in circles? There's more and more bad numbers all over the place.

A. No, in each case the new bad numbers are *simpler* than the original ones; one of them was created earlier. We can't go on and on finding earlier and earlier sets of bad numbers, so there can't be any bad sets at all!

B. (brightening) Oho! What you're saying is this: Each number x was created on some day $d(x)$. If there are three bad numbers (x, y, z), for which the sum of their creation days is $d(x) + d(y) + d(z) = n$, then one of your two cases applies and gives three bad numbers whose day-sum is less than n. Those, in turn, will produce a set whose day-sum is still less and so on; but that's impossible since there are no three numbers whose day-sum is less than 3.

A. Right, the sum of the creation days is a nice way to express the proof. If there are no three bad numbers (x, y, z) whose day-sum is less than n, the two cases show that there are none whose day-sum equals n. I guess it's a proof by induction on the day-sum.

B. You and your fancy words. It's the *idea* that counts.

A. True; but we need a name for the idea, so we can apply it more easily next time.

B. Yes, I suppose there will be a next time . . .

Okay, I guess there's no reason for me to be uptight any more about the New Math jargon. You know it and I know it; we've just proved the *transitive law*.

A. (sigh) Not bad for two amateur mathematicians!

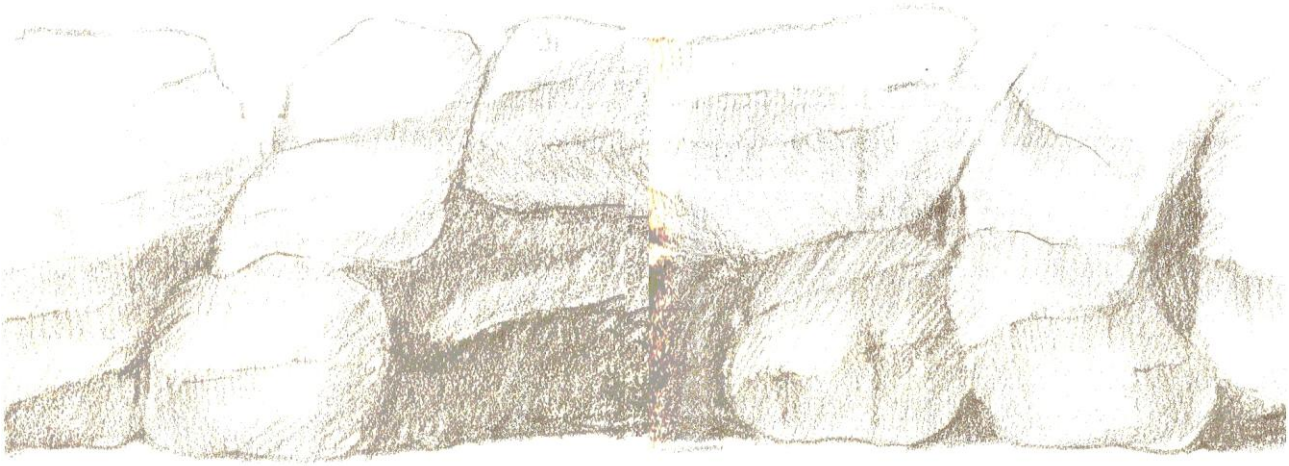
B. It was really your doing. I hereby proclaim that the transitive law (T1) shall be known henceforth as Alice's Theorem.

A. C'mon. I'm sure Conway discovered it long ago.

B. But does that make your efforts any less creative? I bet every great mathematician started by rediscovering a bunch of "well-known" results.

A. Gosh, let's not get carried away dreaming about greatness! Let's just have fun with this.

5 PROGRESS



B. I just thought of something. Could there possibly be two numbers that aren't related to each other at all? I mean

$$x \nleq y \quad \text{and} \quad y \nleq x,$$

like one of them is out of sight or in another dimension or something. It shouldn't happen, but how would we prove it?

A. I suppose we could try the same technique that worked before. If x and y are bad numbers in this sense, then either some $x_L \geq y$ or $x \geq$ some y_R.

B. Hmm. Suppose $y \leq x_L$. Then if $x_L \leq x$, we would have $y \leq x$ by our transitive law, and we have assumed that $y \nleq x$. So $x_L \nleq x$. In the other case, $y_R \leq x$, the same kind of figuring would show that $y \nleq y_R$.

A. Hey, that's very shrewd! All we have to do now to show that such a thing can't happen is prove something I've suspected for a long time. Every number x must lie between all the elements of its sets X_L and X_R. I mean,

$$X_L \leq x \qquad \text{and} \qquad x \leq X_R. \tag{T2}$$

B. That shouldn't be hard to prove. What does $x_L \nleq x$ say?

A. Either there is a number x_{LL} in X_{LL}, with $x_{LL} \geq x$, or else there is a number x_R in X_R with $x_L \geq x_R$. But the second case can't happen, by rule (1).

B. I *knew* we were going to use rule (1) sooner or later. But what can we do with x_{LL}? I don't like double subscripts.

A. Well, x_{LL} is an element of the left set of x_L. Since x_L was created earlier than x, we can at least assume that $x_{LL} \leq x_L$, by induction.

B. Lead on.

A. Let's see, $x_{LL} \leq x_L$ says that $x_{LLL} \ngeq x_L$ and . . .

B. (interrupting) I don't want to look at this—your subscripts are getting worse.

A. You're a big help.

B. Look, I *am* helping, I'm telling you to keep away from those hairy subscripts!

A. But I . . . Okay, you're right, excuse me for going off on such a silly tangent. We have $x \leq x_{LL}$ and $x_{LL} \leq x_L$, so the transitive law tells us that $x \leq x_L$. This probably gets around the need for extra subscripts.

B. Aha, that does it. We can't have $x \leq x_L$, because that would mean $X_L \ngeq x_L$, which is impossible since x_L is one of the elements of X_L.

A. Good point, but how do you know that $x_L \leq x_L$.

B. What? You mean we've come this far and haven't even proved that a number is like itself? Incredible . . . there must be an easy proof.

A. Maybe you can see it, but I don't think it's obvious. At any rate, let's try to prove

$$x \leq x. \tag{T3}$$

This means that $X_L \ngeq x$ and $x \ngeq X_R$.

B. It's curiously like (T2). But uh-oh, here we are in the same spot again, trying to show that $x \leq x_L$ is impossible.

A. This time it's all right, Bill. Your argument shows that $x \leq x_L$ implies $x_L \ngeq x_L$, which is impossible by induction.

B. Beautiful! That means (T3) is true, so everything falls into place. We've got the "$X_L \leq x$" half of (T2) proved, and the other half must follow by the same argument, interchanging left and right everywhere.

A. And like we said before, (T2) is enough to prove that all numbers are related; in other words

$$\text{if} \quad x \nleq y \quad \text{then} \quad y \leq x. \tag{T4}$$

B. Right. Look, now we don't have to bother saying things so

indirectly any more, since "$x \ngeq y$" is exactly the same as "x is less than y."

A. I see, it's the same as "x is less than or like y but not like y." We can now write

$$x < y$$

in place of $x \ngeq y$, and the original rules (1) and (2) look much nicer. I wonder why Conway didn't define things that way? Maybe it's because a third rule would be needed to define what "less than" means, and he probably wanted to keep down the number of rules.

B. I wonder if it's possible to have two different numbers which are like each other. I mean, can we have both $x \leq y$ and $x \geq y$ when X_L is not the same set as Y_L?

A. Sure, we saw something like that before lunch. Don't you remember, we found that $0 \leq y$ and $y \leq 0$ when $y = (\{-1\}, \emptyset)$. And I think $(\{0, 1\}, \emptyset)$ will turn out to be like $(\{1\}, \emptyset)$.

B. You're right. When $x \leq y$ and $x \geq y$, I guess x and y are effectively equal for all practical purposes, because the transitive law tells us that $x \leq z$ if and only if $y \leq z$. They're interchangeable.

A. Another thing, we've also got two more transitive laws, I mean

$$\begin{array}{llllll} \text{if} & x < y & \text{and} & y \leq z & \text{then} & x < z; \quad \text{(T5)} \\ \text{if} & x \leq y & \text{and} & y < z & \text{then} & x < z. \quad \text{(T6)} \end{array}$$

B. Very nice—in fact, these both follow immediately from (T1), if we consider "$x < y$" equivalent to "$x \ngeq y$". There's no need to use (T2), (T3), or (T4) in the proofs of (T5) and (T6).

A. You know, when you look over everything we've proved, it's really very pretty. It's amazing that so much flows out of Conway's two rules.

B. Alice, I'm seeing a new side of you today. You really put to rest the myth that women can't do mathematics.

A. Why, thank you, gallant knight!

B. I know it sounds crazy, but working on this creative stuff with you makes me feel like a stallion! You'd think so much brainwork would turn off any physical desires, but really—I haven't felt quite like this for a long time.

A. To tell the truth, neither have I.

B. Look at that sunset, just like in the poster we bought once. And look at that water.

A. (running) Let's go!

6 THE THIRD DAY

B. Boy, I never slept so well.

A. Me too. It's so great to wake up and be really awake, not just "coffee-awake."

B. Where were we yesterday, before we lost our heads and forgot all about mathematics?

A. (smiling) I think we had just proved that Conway's numbers

behave like all little numbers should; they can be arranged in a line, from smallest to largest, with every number being greater than those to its left and less than all those on its right.

B. Did we really prove that?

A. Yes, anyway at least the unlike numbers keep in line, because of (T4). Every new number created must fall into place among the others.

B. Now it should be pretty easy for us to figure out what happened on the Third Day; those 20×20 calculations must be reduced 'way down. Our theorems (T2) and (T3) show that

so seven of the numbers are placed already and it's just a matter of fitting the other ones in.

You know, now that it's getting easier, this is much more fun than a crossword puzzle.

A. We also know, for example, that

lies somewhere between  and  . Let's check it against the middle element 0.

B. Hmm, it's both $\leq$ and ≥ 0, so it must be like 0, according to rule (3). As I said yesterday, it's effectively equal to 0, so we might as well forget it. That's eight down and twelve to go.

A. Let's try to get rid of those ten cases where X_L or X_R have more than one element, like I tried to do yesterday morning. I had an idea during the night which might work. Suppose

$x = (X_L, X_R)$ is a number, and we take any other sets of numbers Y_L and Y_R, where

$$Y_L < x < Y_R.$$

Then I think it's true that x is like z, where

$$z = (Y_L \cup X_L, X_R \cup Y_R).$$

In other words, enlarging the sets X_L and X_R, by adding numbers on the appropriate sides, doesn't really change x.

B. Let's see, that sounds plausible. At any rate, z is a number, according to rule (1); it will be created sooner or later.

A. In order to show that $z \leq x$, we have to prove that

$$Y_L \cup X_L < x \qquad \text{and} \qquad z < X_R.$$

But that's easy, now, since we know that $Y_L < x$, $X_L < x$, and $z < X_R \cup Y_R$, by (T3).

B. And the same argument, interchanging left and right, shows that $x \leq z$. You're right, it's true:

$$\begin{array}{ll} \text{if} & Y_L < x < Y_R, \\ \text{then} & x \equiv (Y_L \cup X_L, X_R \cup Y_R). \end{array} \qquad \text{(T7)}$$

(I'm going to write "$x \equiv z$," meaning x is like z, I mean $x \leq z$ and $z \leq x$.)

A. That proves just what we want. For example,

and so on.

B. So we're left with only two cases: $\langle - : \rangle$ and $\langle : | \rangle$.

A. Actually, (T7) applies to both of these, too, with $x = 0$!

B. Cle-*ver*. So the Third Day is now completely analyzed; only those seven numbers we listed before are essentially different.

A. I wonder if the same thing won't work for the following days, too. Suppose the different numbers at the end of n days are

$$x_1 < x_2 < \cdots < x_m.$$

Then maybe the only new numbers created on the $(n + 1)$st day will be

$$(\emptyset, \{x_1\}), \quad (\{x_1\}, \{x_2\}), \quad \ldots, \quad (\{x_{m-1}\}, \{x_m\}), \quad (\{x_m\}, \emptyset).$$

B. Alice, you're wonderful! If we can prove this, it will solve infinitely many days in one swoop! You'll get ahead of the Creator himself.

A. But maybe we can't prove it.

B. Anyway let's try some special cases. Like, what if we had the number $(\{x_{i-1}\}, \{x_{i+1}\})$; it would have to be equal to one of the others.

A. Sure, it equals x_i, because of (T7). Look, each element of X_{iL} is $\leq x_{i-1}$, and each element of X_{iR} is $\geq x_{i+1}$. Therefore, by (T7), we have

$$x_i \equiv (\{x_{i-1}\} \cup X_{iL}, X_{iR} \cup \{x_{i+1}\}).$$

And again by (T7),

$$(\{x_{i-1}\}, \{x_{i+1}\}) \equiv (X_{iL} \cup \{x_{i-1}\}, \{x_{i+1}\} \cup X_{iR}).$$

By the transitive law, $x_i \equiv (\{x_{i-1}\}, \{x_{i+1}\})$.

B. (shaking his head) Incredible, Holmes!

A. Elementary, my dear Watson. One simply uses deduction.

B. Your subscripts aren't very nice, but I'll ignore it this time. What would you do with the number $(\{x_{i-1}\}, \{x_{j+1}\})$ if $i < j$?

A. (shrugging her shoulders) I was afraid you'd ask that. I don't know.

B. Your same argument would work beautifully if there was a number x where each element of X_L is $\leq x_{i-1}$ and each element of X_R is $\geq x_{j+1}$.

A. Yes, you're right, I hadn't noticed that. But all those elements $x_i, x_{i+1}, \ldots, x_j$ in between might interfere.

B. I suppose so . . . No, I've got it! Let x be the one of $x_i, x_{i+1}, \ldots, x_j$ which was created *first*. Then X_L and X_R can't involve any of the others! So $(\{x_{i-1}\}, \{x_{j+1}\}) \equiv x$.

A. Allow me to give you a kiss for that.

. . . . . .

. . . . . .

B. (smiling) The problem isn't completely solved, yet; we have to consider numbers like $(\emptyset, \{x_{j+1}\})$ and $(\{x_{i-1}\}, \emptyset)$. But in the first case, we get the first-created number of $x_1, x_2, \ldots, x_j$, and in the second case it's the first-created number of $x_i, x_{i+1}, \ldots, x_m$.

A. What if the first-created number wasn't unique? I mean, what if more than one of the $x_i, \ldots, x_j$ were created on that earliest day?

B. Whoops . . . No, it's okay, that can't happen, because the proof is still valid and it would show that the two numbers are both like each other, which is impossible.

A. Neato! You've solved the problem of all the days at once.

B. With your help. Let's see, on the fourth day there will be 8 new numbers, then on the fifth day there are 16 more, and so on.

A. Yes, after the nth day, exactly $2^n - 1$ different numbers will have been created.

B. You know, I don't think that guy Conway was so smart after all. I mean, he could have just given much simpler rules, with the same effect. There's no need to talk about sets of numbers, and all that nonsense; he simply would have to say that the new numbers are created between existing adjacent ones, or at the ends.

C. **Rubbish. Wait until you get to infinite sets.**

A. What was that? Did you hear something? It sounded like thunder.

B. I'm afraid we'll be getting into the monsoon season pretty soon.